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Closest Bifurcation Analysis and Robust Stability Design of Flexible Satellites

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We consider the design of a satellite to ensure robust stability in a high-dimensional parameter space. Starting with a given nominal design that is stable, we compute instabilities that are locally closest to the nominal design in the design parameter space. If these worst-case instabilities are too close, we compute a design that is sufficiently far from the worst-case instabilities and hence sufficiently robust. The methods are based on computations from bifurcation theory and are considerably simplified by our assumption of a Hamiltonian satellite, which includes spin-stabilized satellites, dual-spin-stabilized satellites, and gravity-gradient-stabilized satellites. The Hamiltonian assumption is then relaxed to allow the inclusion of damping terms. We illustrate the computations by ensuring the robust stability of a flexible dual-spin satellite with six design parameters.

Introduction

LET λ be a vector of satellite design parameters such as masses, moments of inertia, and lengths and stiffnesses of flexible components. Then the parameter vector λ needs to be designed so that the satellite is robustly stable. That is, given a nominal design λ_0 that is stable, the stability of designs near λ_0 in parameter space should be ensured. Although this process is straightforward if only one or two parameters vary, satellite designs are typically described by many parameters, and it is not usually clear which subsets or combinations of parameters should be varied in order to check and ensure robust stability of the design. That is, the high dimension of the parameter space is typically an obstacle in design. We see our methods as contributing a tool that gives quantitative help in ensuring robust stability of a satellite design in higher dimensional parameter spaces.

In general, a bifurcation occurs when there is a qualitative change in the system dynamics as explained in the introductory texts.[1–4] The qualitative change we are interested in is the stability of the equilibrium at which the satellite is to be operated. A bifurcation of this equilibrium is the condition of marginal stability through which the equilibrium must pass if quasistatic variations in satellite parameters cause the equilibrium to lose stability. We write λ_* for the critical parameter values at which the system bifurcates and the equilibrium stability is lost. The robust stability problem is to ensure that the design parameters are sufficiently far from the critical parameter values corresponding to bifurcations and instability. This paper computes critical parameter values λ_* that are locally closest in parameter space to a given nominal design λ_0. The computations are based on both standard[1] and novel[5] computations in bifurcation theory and are simplified considerably by the assumption of a Hamiltonian satellite. The general study of instabilities of mechanical systems as bifurcation instabilities with several parameters is familiar from works such as the text by Huseyin[2] and the paper by Plaut[6] on the stability boundaries of multiply loaded shallow arches.

We model a satellite as a Hamiltonian system with a vector $\lambda \in \boldsymbol{R}^m$ of design parameters. The attitude of the satellite is specified by a state vector of generalized coordinates, and the origin is assumed to be an equilibrium for all values of the design parameters λ. We assume a nominal design λ_0 at which the origin is stable, and we assume that stability is established by showing that the system Hamiltonian is positive definite at λ_0 and invoking Lyapunov's direct method (Ref. 7, p. 244). As the parameters λ vary from the nominal parameters λ_0, the origin can lose stability in a bifurcation as the Jacobian of the system at the origin becomes singular, and we denote the set of such λ in the parameter space $\boldsymbol{R}^m$ by Σ. Here, Σ is a boundary in parameter space of the designs that are stable, and the designer seeks to maintain a sufficient distance from Σ to ensure that the design is robustly stable. A good choice of the sufficient distance requires engineering judgment of design safety factors and likely parameter variations and tolerances.

Typically Σ consists of hypersurfaces in $\boldsymbol{R}^m$ and their intersections. Robust stability of the design can be addressed by monitoring the position of the nominal design λ_0 relative to Σ and correcting the design if λ_0 is too close to Σ. In particular, it is useful to calculate critical parameters λ_* in Σ for which $|\lambda_* - \lambda_0|$ is a local minimum of the distance from λ_0 to Σ. Then the line segment $\lambda_0\lambda_*$ represents a worst-case parameter variation and $|\lambda_* - \lambda_0|$ is a stability margin measuring the proximity to bifurcation. We call the bifurcation at λ_* "a closest bifurcation" with the understanding that the distance to bifurcation is measured in parameter space relative to the fixed value λ_0.

This paper explains how to compute the stability margin $|\lambda_* - \lambda_0|$ and how to use the sensitivity of $|\lambda_* - \lambda_0|$ to change the nominal design λ_0 to increase $|\lambda_* - \lambda_0|$ if $|\lambda_* - \lambda_0|$ is too small. After illustrating the computation with a simple rigid-satellite example, we use the computations to design a flexible dual-spin satellite with multiple parameters so that a robustness criterion is met. The computations apply more generally to Hamiltonian systems with additional damping terms provided these terms have no linear dependence on position. That is, the computations apply provided the Taylor series expansion of the damping about the equilibrium point has zero coefficients for the linear position terms. A previous version of this work appeared in the conference paper.[8]

The focus of this paper on Hamiltonian satellites is motivated by an interest in flexible satellites stabilized by "passive" means such as spin stabilization, dual-spin stabilization, and gravity-gradient stabilization. Our intent is to add a useful design tool to the literature on passive stabilization methods (for recent studies, see (Refs. 9–15). As stated above, the methods presented here work in the presence of additional damping terms provided these terms have no linear dependence on position, e.g., in the presence of Rayleigh damping.

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The methods of this paper are generally not appropriate for satellites with active control systems because the Hamiltonian structure is usually destroyed and the generic bifurcations associated with loss of equilibrium stability may differ. However, it would be interesting in future work to try to apply the closest bifurcation methods appropriate to non-Hamiltonian systems[5] to the robust parameter space design of satellites with active control.

Closest bifurcation analysis has also been applied to voltage collapse in electric power systems by Dobson and Lu.[16] In this application, the parameters are the system power loadings, and critical loadings λ_* correspond to saddle node bifurcation, voltage collapse, and blackout of the system. Computing the saddle node bifurcation closest in load power parameter space gives a worst-case measure $|\lambda_* - \lambda_0|$ of proximity to voltage collapse, and if this measure is too small, corrective control action can be computed to move the system further from voltage collapse. This monitoring and preventative control of voltage collapse shares the overall approach of this paper but applies it to find controls to steer a power system away from bifurcation rather than refine a satellite design to improve its robustness.

Transcritical and Pitchfork Bifurcations

We describe the nature of the bifurcation instabilities typically encountered when the Jacobian of the system at the origin is singular. Let the equations of motion of the system be

$$\dot{x} = f(x, \lambda), \qquad x \in \boldsymbol{R}^n, \qquad \lambda \in \boldsymbol{R}^m \tag{1}$$

where f is a C^∞ smooth function with $f(0, \lambda) = 0$ for all λ so that the origin is always an equilibrium. Write Df or $Df|_{(0,\lambda)}$ for the Jacobian of f with respect to x evaluated at the origin and at parameters λ. At critical parameters $\lambda_* \in \Sigma$, Df has a zero eigenvalue and a transcritical or pitchfork bifurcation occurs. This zero eigenvalue is generically unique. Note that we have eliminated the possibility of a Hopf bifurcation via our assumption that the system Hamiltonian is positive definite at λ_0. Stability can then only be lost when the system Hamiltonian ceases to be positive definite, and this requires the passage of an eigenvalue through the origin as discussed later. Let the left and right eigenvectors of the Jacobian corresponding to the zero eigenvalue be w'_* and v'_*, respectively. Moreover, in the absence of special symmetries in the problem, the bifurcation will generically be a transcritical bifurcation satisfying the transversality condition

$$w'_* D_{x\lambda} f|_{(0,\lambda_*)} v'_* \neq 0 \tag{2}$$

and an additional second-order transversality condition.[4] [$D_{x\lambda} f$ is the second derivative $\partial^2 f/(\partial\lambda\, \partial x)$.] If the system has the odd symmetry $f(-x) = -x$ then the bifurcation will generically be a pitchfork bifurcation satisfying (2) and an additional third order transversality condition.[4] In either case, Σ is a hypersurface in a neighborhood of λ_*, and a normal vector to the hypersurface at λ_* is given by the transversality condition (2):

$$N(\lambda_*) = w'_* D_{x\lambda} f|_{(0,\lambda_*)} v'_* \tag{3}$$

A standard argument proving Eq. (3) is given in the Appendix.

Computing Bifurcation for Given Parameter Change

We briefly review computing transcritical or pitchfork bifurcations for a given parameter change and state a direct method for the computation. Suppose we specify a particular direction of parameter change from the nominal design λ_0. That is, we specify a ray in parameter space based at λ_0 with a unit vector n_0 so that the parameters λ along the ray are given by

$$\lambda = \lambda_0 + l' n_0 \tag{4}$$

as the factor l' assumes positive real values. There are several methods to compute the closest bifurcation assuming this ray of parameter change. That is, we can compute a critical factor l so that $\lambda_1 = \lambda_0 + l n_0 \in \Sigma$. Since n_0 is a unit vector, $l = |\lambda_1 - \lambda_0|$. Here, l is a stability margin along a given direction of parameter change. This computation can be done by using continuation methods, direct methods, or optimization methods. The text by Seydel[1] gives an entry to the extensive numerical analysis literature on continuation and direct methods.

A direct method in which solution of the following equations by Newton's method yields the stability margin l and the right eigenvector v'_* is

$$Df|_{(0,\lambda_0 + l n_0)} v'_* = 0 \tag{5}$$

$$c^T v'_* = 1 \tag{6}$$

Equation (5) states that the Jacobian Df evaluated at the bifurcation is singular with right eigenvector v'_*. The term $c \in \boldsymbol{R}^p$ is a fixed vector and Eq. (6) ensures that the right eigenvector v'_* is nonzero. Continuation methods are also a good choice for computing l.

Simplifications for Hamiltonian Systems

The previous sections characterize the critical parameters Σ as those parameters at which the Jacobian of f is singular;

$$\Sigma = \{\lambda \mid Df|_{(0,\lambda)} \quad \text{has a zero eigenvalue}\} \tag{7}$$

and state a formula for the normal vector to Σ and a method for computing points on Σ based on this characterization. In this section we give alternative characterizations of Σ for Hamiltonian systems (see also Ref. 17) that lead to a formula for the normal vector to Σ and a method for computing points on Σ that are simpler.

Consider a parameterized, smooth, holonomic Hamiltonian system with Hamiltonian H and the origin as an equilibrium:

$$(\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right) = f(q, p, \lambda) \tag{8}$$

Write $D^2 H(\lambda)$ for the Hessian matrix of H evaluated at the origin. The Jacobian and the Hessian are related by the identity

$$Df = J D^2 H \tag{9}$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

In particular,

$$\det Df = \det D^2 H \tag{10}$$

since $\det J = 1$ so that

$$\Sigma = \{\lambda \mid D^2 H(\lambda) \text{ has a zero eigenvalue}\} \tag{11}$$

Note that the Hamiltonian H is expressed as a function of the Hamiltonian coordinates q, p. It is often easier to obtain the Hamiltonian expressed as a function h of the Lagrange coordinates q, $\dot{q}$, and we now deduce the corresponding result for h. Let the transformation between Lagrangian and Hamiltonian coordinates be given by $P = T(Q)$. Applying the chain rule for differentiation yields

$$D^2 h = D(DHDT) = (DT)^T D^2 H DT + DH D^2 T \tag{12}$$

and, since DH vanishes at the equilibrium,

$$D^2 h = (DT)^T D^2 H DT \tag{13}$$

The conjugacy, Eq. (13), implies that $D^2 h$ and $D^2 H$ have the same inertia[18] and in particular that $D^2 h$ and $D^2 H$ have the same number of zero eigenvalues. Hence

$$\Sigma = \{\lambda \mid D^2 h(\lambda) \text{ has a zero eigenvalue}\} \tag{14}$$

Now suppose that the Hamiltonian has the form

$$h = h_2 + h_0 \tag{15}$$

where h_2 is a positive-definite quadratic form in the generalized velocities $\dot{q}$ and h_0 is a function only of the generalized coordinates q. Then

$$D^2h = \begin{bmatrix} D_q^2 h_0 & 0 \\ 0 & D_{q\text{dot}}^2 h_2 \end{bmatrix} \tag{16}$$

where D_q denotes a derivative with respect to q and $D_{q\text{dot}}$ denotes a derivative with respect to $\dot{q}$. Then

$$\det D^2h = \det D_q^2 h_0 \det D_{q\text{dot}}^2 h_2 \tag{17}$$

and since h_2 is a positive-definite quadratic form, $\det D_{q\text{dot}}^2 h_2(\lambda) > 0$ for all λ and

$$\Sigma = \{\lambda \mid D_q^2 h_0(\lambda) \text{ has a zero eigenvalue}\} \tag{18}$$

Suppose there is a generic transcritical or pitchfork bifurcation at parameters $\lambda_* \in \Sigma$. Then the zero eigenvalue of $D_q^2 h_0$ is unique (see Appendix) and we write v_* for the right eigenvector corresponding to the zero eigenvalue. Since $D_q^2 h_0$ is symmetric, the left eigenvector corresponding to the zero eigenvalue is v_*^T. The following formula for the normal vector to Σ at λ_* is analogous to Eq. (3) and is derived in the Appendix:

$$N(\lambda_*) = v_*^T D_\lambda D_q^2 h_0(\lambda_*) v_* \tag{19}$$

Given the direction of parameter change n_0, a direct method to compute the stability margin l and the right eigenvector v_* is

$$D_q^2 h_0(\lambda_0 + l n_0) v_* = 0 \tag{20}$$

$$c^T v_* = 1 \tag{21}$$

Equations (19–21) are much simpler to compute for Hamiltonian systems of the form of Eq. (15) because they are expressed in terms of a submatrix of the Hamiltonian in Lagrangian coordinates. There are analogous equations for Hamiltonian systems not of the form of Eq. (15); simply replace $D_q^2 h_0$ by $D^2 h$ in Eqs. (19–21).

Hamiltonian Systems, Σ, and Lyapunov's Direct Method

The introduction states that stability at the nominal design λ_0 was established by showing that the system Hamiltonian is positive definite at λ_0 and invoking Lyapunov's direct method. This is important because it guarantees that stability can only be lost by crossing Σ; in other words, it guarantees that no bifurcations can occur before Σ is reached. To see this, note that if the system Hamiltonian is positive definite, then the Hessian matrix D^2h is positive definite. Being real, symmetric, and positive definite, D^2h will thus have all of its eigenvalues lying on the positive real axis. Therefore positive definiteness of D^2h, and hence stability, cannot be lost until one of these eigenvalues passes through zero. But, as shown in the preceding section, a zero eigenvalue of D^2h can only occur on Σ.

Iterative Method

The iterative method from Ref. 5 to compute a closest bifurcation has two main ingredients: the formula for the normal vector N to Σ and any of the standard methods for finding the stability margin l assuming a direction of parameter change. The computation of l and N may be iterated to compute the direction n_* and parameter value λ_* of a locally closest transcritical or pitchfork bifurcation and hence the stability margin $|\lambda_* - \lambda_0|$. The procedure is as follows:

1) Let n_0 be an initial guess for the direction n_*.

2) Given n_{i-1}, compute the bifurcation along the ray given by n_{i-1}; that is, compute l_i, λ_i, x_i so that $\lambda_i = \lambda_0 + n_{i-1} l_i \in \Sigma$.

3) Compute the normal vector $N(\lambda_i)$.

4)Set $n_i = \pm N(\lambda_i)/|N(\lambda_i)|$. The sign is chosen so that n_i points outward from the region of stability.

5) Iterate steps 2–4 until convergence of n_i to a value n_*. Then $\lambda_* = \lambda_0 + l_* n_*$.

The direction n_* of a locally closest bifurcation is parallel to the normal vector $N(\lambda_*)$ of Σ at λ_*, and it follows that n_* is a fixed point of the iteration. The quickest way to grasp how the iteration works is to try it with pencil and paper in the case of Σ an ellipse and λ_0 an interior point of the ellipse. Note that the iteration converges in one step if Σ is a hyperplane.

The iteration can be understood as minimizing $|\lambda_* - \lambda_0|$ on a series of tangent hyperplane approximations to Σ. At each iteration, $n_i = N(\lambda_i)$ indicates the direction of the point closest to λ_0 on the tangent hyperplane $T\Sigma_{\lambda_i}$ to Σ at λ_i. The following claims are proved in Ref. 5

1) If the iteration converges exponentially to a fixed point n_* then the parameter $\lambda_* = \lambda_0 + n_* l_*$ specifies a locally closest bifurcation.

2) If λ_* is the parameter of a locally closest bifurcation and Σ is convex or "not too concave" at λ_*, then the direction $n_* = N(\lambda_*)$ is an exponentially stable fixed point of the iteration. (The precise meaning of "not too concave" is that the minimum principal curvature of Σ at λ_* must exceed $-|\lambda_* - \lambda_0|^{-1}$.)

Note that when the iteration converges, it converges to a locally closest bifurcation that is not necessarily a globally closest bifurcation. This is a potential problem in practice, particularly if the hypersurfaces of Σ are corrugated or if λ_0 is close to several portions of Σ.

An initial ray direction n_0 for the iterative method may often be calculated as follows from information available at the nominal design λ_0 (Ref. 16): If λ_0 is close enough to one of the hypersurfaces of Σ, then the eigenvalue of $D_q^2 h_0(\lambda_0)$ of smallest absolute value is the eigenvalue that will become zero as λ_0 moves toward $\lambda_* \in \Sigma$. The corresponding right eigenvector v_0 of $D_q^2 h_0(\lambda_0)$ approximates the right eigenvector v_* at λ_* so that $n_0 = v_0^T D_\lambda D_q^2 h_0(\lambda_0) v_0$ approximates n_*. Note that this argument only works when λ_* is close enough to exactly one of the hypersurfaces of Σ; obtaining justifiably good estimates for n_0 in general is an open problem. However, the iterative method seems robust to the choice of n_0. Dobson[16,5] also gives a direct method for computing a closest saddle node bifurcation. That is, equations are given that have solutions that include closest bifurcations. Analogous equations for the closest transcritical or pitchfork bifurcation are straightforward to obtain[5] and can easily be simplified by the Hamiltonian assumptions. However, the solutions of the direct-method equations require the curvature of Σ to be computed in order to confirm that the solution corresponds to a minimum and not just a turning point of the distance $|\lambda_* - \lambda_0|$. There is a formula for the curvature of the saddle node bifurcation set,[5] but no such formula is known for other bifurcations. The difficulty of confirming that the solution is a minimum and the expected superior robustness properties of the iterative method led us to choose the iterative method here. The iterative method only converges to locally closest bifurcations and does not require the curvature of Σ at the solution to be checked.

Sensitivity Formulas and Improving Design Robustness

If the nominal design λ_0 is too close to Σ, then the nominal design should be changed to increase the distance to Σ. An optimal combination of parameters to change can be obtained from the sensitivity to λ_0 of the distance to Σ. Sensitivity formulas are given both for the distance to Σ assuming a given direction of parameter change and for the closest distance to Σ.

Assume the direction of parameter change n_0 so that $\lambda_1 = \lambda_0 + l n_0 \in \Sigma$. Then the sensitivity to λ_0 of the stability margin l is

$$D_{\lambda_0} l = -N(\lambda_1)[N(\lambda_1) n_0]^{-1} \tag{22}$$

That is, the optimum direction of first-order change in λ_0 to increase l is along $-N(\lambda_1)$. Equation (22) is proved in Ref. 19

Let λ_* be a closest bifurcation to λ_0. Then the sensitivity to λ_0 of the stability margin $|\lambda_* - \lambda_0|$ is

$$D_{\lambda_0} |\lambda_* - \lambda_0| = -N(\lambda_*) \tag{23}$$

as proved in Ref. 5. That is, the optimum direction of first-order change in λ_0 to increase $|\lambda_* - \lambda_0|$ is along $-N(\lambda_*)$.

Thus the normal vector $-N(\lambda_*)$ to Σ yields the first-order sensitivity of distance measures to changes in the nominal design. This fact is obvious upon locally approximating Σ by its tangent plane, as explained in Ref. 16. The larger entries in $-N(\lambda_*)$ indicate the

parameters that are most influential on $|\lambda_* - \lambda_0|$ and the proximity of λ_0 to Σ.

Equation (22) can also be applied to determine the sensitivity of the distance $|\lambda_* - \lambda_0|$ to a closest bifurcation to model parameters not initially included in the design parameter space.[19] This is done by augmenting the parameter space with additional model parameters and then computing the normal vector at λ_* in the augmented parameter space using Eq. (19). (Note that the right eigenvector v_* need not be recomputed.)

Suppose that λ_0^0 is an initial nominal design and a minimum stability margin ρ has been selected. If λ_*^0 is a closest bifurcation to λ_0^0 and $|\lambda_*^0 - \lambda_0^0| < \rho$, then we want to change the design to λ_0^1 to satisfy the minimum requirement $|\lambda_*^1 - \lambda_0^1| = \rho$. However, it is also desirable to minimize the design change $|\lambda_0^1 - \lambda_0^0|$ since we assume that λ_0^0 has already been optimized with respect to design criteria other than robust stability. A simple approximate method of reaching these objectives is to move λ_0^0 along the normal vector $-N(\lambda_*)$ to make up the balance of the distance ρ to obtain the design

$$\lambda_0^1 = \lambda_0^0 - \frac{N(\lambda_*)}{|N(\lambda_*)|}\left(\rho - |\lambda_*^0 - \lambda_0^0|\right) \tag{24}$$

The sensitivity formula, Eq. (23), ensures that the design change $|\lambda_0^1 - \lambda_0^0|$ is minimized to a first-order approximation. By construction, $|\lambda_*^0 - \lambda_0^1| = \rho$, but it is necessary to check that λ_*^0 is a bifurcation closest to λ_0^1, as we could have moved closer to a different point on the bifurcation surface when we changed the design from λ_0^0 to λ_0^1. That is, the closest bifurcation algorithm should be rerun with nominal design λ_0^1 to obtain a closest bifurcation λ_*^1. If $\lambda_*^1 = \lambda_*^0$, then λ_0^1 gives a new design that satisfies the robustness criterion. If $\lambda_*^1 \neq \lambda_*^0$ and $|\lambda_*^1 - \lambda_0^1| < \rho$, then a further iteration to find λ_0^2 is necessary.

Since, for a closest bifurcation, $\lambda_*^0 - \lambda_0^0$ is parallel to $N(\lambda_*)$, Eq. (24) can be rewritten as

$$\lambda_0^1 = \lambda_0^0 - \frac{\lambda_*^0 - \lambda_0^0}{|\lambda_*^0 - \lambda_0^0|}\left(\rho - |\lambda_*^0 - \lambda_0^0|\right) \tag{25}$$

Nonconservative Systems

This section shows that the computations for Hamiltonian systems developed in previous sections generalize to systems with additional damping terms provided these terms have no linear dependence on position. Indeed, the critical parameter set Σ and the closest bifurcations are independent of the presence of this type of damping. Recalling from previous sections that stability of our Hamiltonian system was determined via Lyapunov's direct method, we note that $\dot{H}$ will still be negative semidefinite in the presence of damping, so that stability of the origin will be preserved. We now discuss the effect of damping on the bifurcation set Σ.

Consider a holonomic, conservative system with Hamiltonian H. Lagrange's equations for the conservative system are

$$\ddot{q} = F(q, \dot{q}, \lambda) \tag{26}$$

We also consider a holonomic nonconservative system obtained by adding a term $C(q, \dot{q}, \lambda)$ satisfying $C_q = 0$ to Lagrange's equations

$$\ddot{q} = F_{nc}(q, \dot{q}, \lambda) = F(q, \dot{q}, \lambda) + C(q, \dot{q}, \lambda) \tag{27}$$

First note that, if $y = \dot{q}$, then system (27) can be written as the first-order equations

$$(\dot{q}, \dot{y}) = (y, F(q, y, \lambda) + C(q, y, \lambda)) = f(q, y, \lambda) \tag{28}$$

and the Jacobian Df of the state-space form of Lagrange's equations takes the form

$$Df = \begin{bmatrix} 0 & I \\ F_q & F_y + C_y \end{bmatrix} \tag{29}$$

Apply the Laplace expansion of the determinant[20] to obtain

$$\det Df = -\det F_q \tag{30}$$

Similar observations for a class of second-order systems appear in Refs. 2 and 21. It is clear from Eq. (30) that $\det Df$ is independent of C and in particular that $\det Df$ is identical to the determinant of the Jacobian of the conservative system (26). Since Σ is defined by the vanishing of Df, Σ and the closest bifurcation are independent of the nonconservative forces.

Satellite Model

The satellite considered in this paper is shown in Fig. 1 and consists of a central body containing an internal rotor and two rectangular solar panels supported by massless, elastic shafts with circular cross sections. The central body, rotor, and panels are assumed to be rigid. Each supporting shaft has polar moment of inertia J, modulus of rigidity G, and length L. A guy-wire system is assumed to constrain panel-shaft vibrations to purely torsional modes.[12,22,23]

The central body has principal body axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ that are also principal body axes for the entire satellite. The origin of the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coordinate system is the center of mass of the central body and the center of mass for the entire satellite. The principal moments of inertia of the panels about their principal axes are denoted A_p, B_p, and C_p and the principal axes of the panels are aligned with the axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, as shown in Fig. 1. The center of mass of each panel lies along the $\boldsymbol{j}$ axis and is located a distance l from the center of mass of the central body. Generalized coordinates α_1 and α_2 describe the angle each panel makes with the $\boldsymbol{j}, \boldsymbol{k}$ plane and are also used to measure the amount of twist of each shaft. The center of mass of the rotor is located at O and the rotor spins about the $\boldsymbol{k}$ axis at a constant rate of ω_s relative to the central body.

The principal moments of inertia of the entire satellite in its undeformed state about the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ axes are given by

$$A = A_c + A_r + 2A_p + 2m_p l^2$$

$$B = B_c + A_r + 2B_p$$

$$C = C_c + C_r + 2C_p + 2m_p l^2$$

The center of mass of the satellite is assumed to move in a circular orbit, with orbital angular speed Ω and radial distance R_0, about the center of a spherically symmetric planet. An equilibrium position is defined to be one in which the satellite is at rest with respect to the set of orbital axes $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ shown in Fig. 2. The orbital axes $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are defined as follows: 1) the origin is at the center of

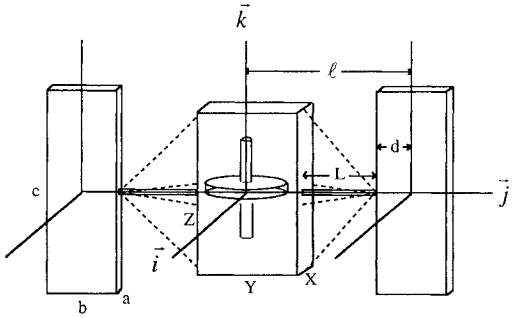

Fig. 1 Satellite model.

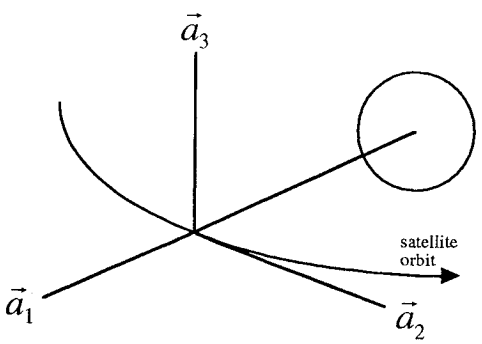

Fig. 2 Orbital axes.

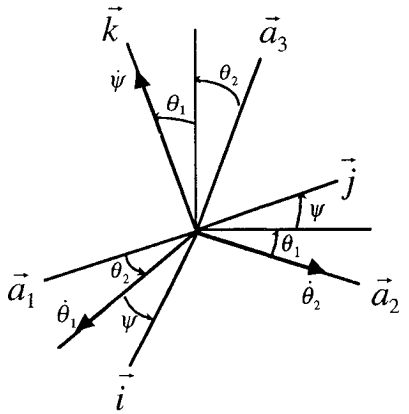

Fig. 3 Rotation sequence for i,j,k axes.

mass of the satellite, 2) a_1 is in the radial direction, 3) a_2 is tangent to the orbit in the direction of motion, and 4) a_3 is normal to the orbit plane. The axes i,j,k of the satellite are oriented relative to the orbital axes by a 2-1-3 rotation sequence through θ_2, θ_1, and ψ, respectively (Fig. 3). It is shown in Ref. 7 that $\theta_2 = \theta_1 = \psi = 0$ is an equilibrium position of the satellite, and this is the equilibrium position used in this research.

Damping

As discussed in the section on nonconservative systems, the stability boundary Σ is independent of damping terms that have no linear dependence on position. Therefore damping of this type can be added to the satellite model described in what follows without affecting any of closest bifurcation results calculated for the satellite.

Energy Expressions

The expression for the kinetic energy of the satellite is given by

$$T = T_{\text{central body}} + T_{\text{rotor}} + T_{\text{panel 1}} + T_{\text{panel 2}}$$

where

$$T_{\text{central body}} = \tfrac{1}{2}M_c v_c \cdot v_c + \tfrac{1}{2}\omega_c \cdot I_c \cdot \omega_c$$

$$T_{\text{rotor}} = \tfrac{1}{2}m_r v_r \cdot v_r + \tfrac{1}{2}\omega_r \cdot I_r \cdot \omega_r$$

$$T_{\text{panel 1}} = \tfrac{1}{2}m_p v_{p1} \cdot v_{p1} + \tfrac{1}{2}\omega_{p1} \cdot I_{p1} \cdot \omega_{p1}$$

$$T_{\text{panel 2}} = \tfrac{1}{2}m_p v_{p2} \cdot v_{p2} + \tfrac{1}{2}\omega_{p2} \cdot I_{p2} \cdot \omega_{p2}$$

For the central body, the rotor, panel 1, and panel 2, we denote the velocities of the centers of mass by v_c, v_r, v_{p1}, v_{p2}, the angular velocities by $\omega_c, \omega_r, \omega_{p1}, \omega_{p2}$, and the moment of inertia tensors by I_c, I_r, I_{p1}, I_{p2}.

The potential energy consists of the gravitational potential plus the strain energy of the shafts:

$$V = V_{\text{grav}} + V_{\text{strain 1}} + V_{\text{strain 2}}$$

where

$$V_{\text{grav}} = \left(\frac{-\Omega^2}{4}\right)\Big[\big(3l_{11}^2 - 1\big)(I_{yy} + I_{zz} - I_{xx}) + \big(3l_{21}^2 - 1\big)(I_{xx} + I_{zz} - I_{yy}) + \big(3l_{31}^2 - 1\big)(I_{xx} + I_{yy} - I_{zz}) + 12(l_{11}l_{21}I_{xy} + l_{11}l_{31}I_{xz} + l_{21}l_{31}I_{yz})\Big]$$

$$V_{\text{strain 1}} = \frac{JG\alpha_1^2}{2L}$$

$$V_{\text{strain 2}} = \frac{JG\alpha_2^2}{2L}$$

and

$$l_{11} = \cos\theta_2 \cos\psi + \sin\theta_2 \sin\theta_1 \sin\psi$$

$$l_{21} = -\cos\theta_2 \sin\psi + \sin\theta_2 \sin\theta_1 \cos\psi$$

$$l_{31} = \sin\theta_2 \cos\theta_1$$

and $I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{xz}$, and I_{yz} are the instantaneous moments of inertia of the entire satellite about the axes i,j,k given by

$$I_{xx} = A_c + A_r + 2m_p l^2 + A_p\big(\cos^2\alpha_1 + \cos^2\alpha_2\big) + C_p\big(\sin^2\alpha_1 + \sin^2\alpha_2\big)$$

$$I_{yy} = B_c + A_r + 2B_p$$

$$I_{zz} = C_c + C_r + 2m_p l^2 + A_p\big(\sin^2\alpha_1 + \sin^2\alpha_2\big) + C_p\big(\cos^2\alpha_1 + \cos^2\alpha_2\big)$$

$$I_{xy} = I_{yz} = 0$$

$$I_{xz} = (A_p - C_p)(\sin\alpha_1 \cos\alpha_1 + \sin\alpha_2 \cos\alpha_2)$$

By constructing the Hamiltonian of the system from the above energy expressions, it can be shown that the system Hamiltonian is of the form $h = h_2 + h_0$ of Eq. (15).

Hessian Matrix

As discussed in the satellite model section, the generalized coordinates for the satellite are

$$(q_1, q_2, q_3, q_4, q_5) = (\theta_1, \theta_2, \psi, \alpha_1, \alpha_2)$$

Evaluate D^2h_0 for the satellite at the equilibrium point $(q_1, q_2, q_3, q_4, q_5) = (0, 0, 0, 0, 0)$ to obtain

$$D^2h_0 = \begin{bmatrix} h_{11} & 0 & 0 & 0 & 0 \\ 0 & h_{22} & 0 & h_{24} & -h_{24} \\ 0 & 0 & h_{33} & 0 & 0 \\ 0 & h_{24} & 0 & h_{44} & 0 \\ 0 & -h_{24} & 0 & 0 & h_{44} \end{bmatrix}$$

where

$$h_{11} = \big(C_c + C_r + 2C_p + 2m_p l^2 - B_c - A_r - 2B_p\big)\Omega^2 + C_r\omega_s\Omega$$

$$h_{22} = 4(C_c + C_r + 2C_p - A_c - A_r - 2A_p)\Omega^2 + C_r\omega_s\Omega$$

$$h_{24} = 4(A_p - C_p)\Omega^2$$

$$h_{33} = 3\big(B_c + 2B_p - A_c - 2A_p - 2m_p l^2\big)\Omega^2$$

$$h_{44} = \frac{JG}{L} - 4(A_p - C_p)\Omega^2$$

Closest Bifurcation Analysis

Simple Illustrative Example

To first illustrate the closest bifurcation techniques in the simplest way, we apply them to a rigid satellite for which the stability boundaries are known and can be drawn in two dimensions. Define dimensionless parameters $\alpha = (C-B)/mk^2$ and $\beta = (C-A)/mk^2$, where m is the mass of the satellite and k is the radius of gyration of the satellite about the k axis. Then it follows from Ref. 24 that in quadrant I of Fig. 4 the lines $\beta = \alpha$ and $\alpha = 0$ are exact stability boundaries; i.e., the satellite is stable provided the satellite parameter values lie in region Ib and the satellite becomes unstable if its parameter values cross into region Ia or quadrant II. For a rigid satellite, the generalized coordinates are

$$(q_1, q_2, q_3) = (\theta_1, \theta_2, \psi)$$

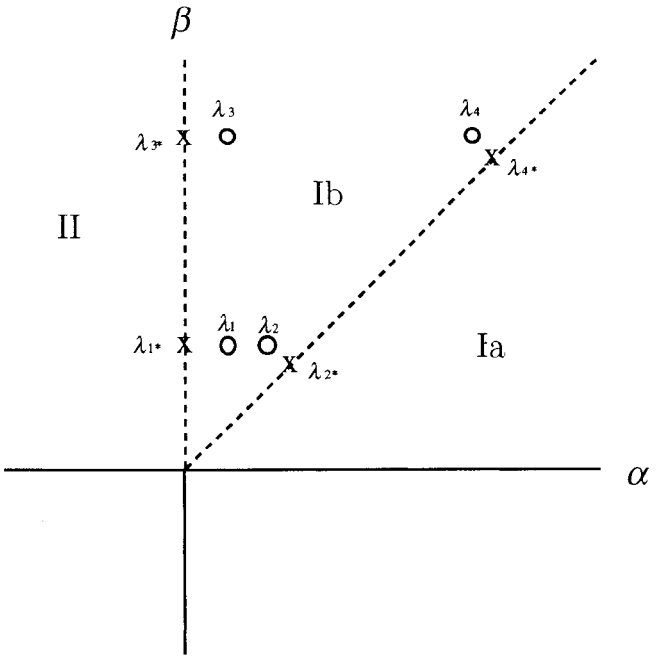

Fig. 4 Parameter space for rigid satellite.

and evaluating D^2h_0 for the satellite at the equilibrium point $(q_1, q_2, q_3) = (0, 0, 0)$, we obtain

$$D^2h_0 = mk^2\Omega^2 \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 4\beta & 0 \\ 0 & 0 & 3(\beta - \alpha) \end{bmatrix}$$

We now choose several points, $\lambda = (\alpha, \beta)$, in stable region Ib and use the methods of this paper to calculate the closest bifurcation, $\lambda_* = (\alpha_*, \beta_*)$, to each point. Choosing $\lambda_1 = (1, 3)$, the closest bifurcation is computed as $\lambda_{1*} = (0, 3)$; for $\lambda_2 = (2, 3)$ the closest bifurcation is computed as $\lambda_{2*} = (2.5, 2.5)$; for $\lambda_3 = (1, 8)$ the closest bifurcation is computed as $\lambda_{3*} = (0, 8)$; and for $\lambda_4 = (7, 8)$ the closest bifurcation is computed as $\lambda_{4*} = (7.5, 7.5)$. As is easily seen by referring to Fig. 4, the methods of this paper do indeed find the closest bifurcation points to each of the points chosen.

Multiparameter Design Example

We use the closest bifurcation computation to design a flexible dual-spin satellite with six parameters so that a robustness criterion is met. Consider a satellite whose central body is a box with dimensions X, Y, and Z in the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ directions. The dimensions of each panel are taken to be a, b, c, as shown in Fig. 1. The following dimensionless parameters are introduced:

$$\tilde{X} \equiv X/Y, \qquad \tilde{Y} \equiv Y/Y = 1, \qquad \tilde{Z} \equiv Z/Y$$

$$\tilde{a} \equiv a/Y, \qquad \tilde{b} \equiv b/Y, \qquad \tilde{c} \equiv c/Y$$

$$\tilde{L} \equiv L/Y, \qquad \tilde{r} \equiv r/Y, \qquad \tilde{h} \equiv h/Y$$

$$\tilde{\omega}_s \equiv \omega_s/(0.0010855\Omega) = \text{dimensionless spin}$$

$$\widetilde{JG} \equiv JG/\left(m_p\Omega^2Y^3\right) = \text{dimensionless stiffness}$$

$$M_c/m_p = \text{ratio of central body mass to panel mass}$$

$$m_r/m_p = \text{ratio of rotor mass to panel mass}$$

We now consider a satellite design problem with six parameters; i.e., we set $\tilde{Y} = 1, \tilde{r} = \frac{1}{2}, \tilde{h} = \frac{1}{4}, \tilde{\omega}_s = 25, \widetilde{JG} = 100$ and $M_c/m_p = 60, m_r/m_p = 16$ and consider the remaining six parameters as design parameters. The design parameters are denoted by the vector $\lambda = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{L}, \tilde{X}, \tilde{Z})$. In addition to requiring that the satellite be stable, we assume a robustness criterion is given in terms of minimum distance in parameter space to the closest bifurcation point. We start with an initial nominal design λ_0^0 and calculate the distance and direction of a closest bifurcation and use these results to move to a new design λ_0^1 satisfying the robustness criterion.

Statement of Design Problem

We wish to design a dual-spin satellite whose parameter vector λ_0 is close to the nominal design

$$\lambda_0^0 = (4.0, 4.0, 0.05, 3.0, 6.1, 6.0)$$

and meets the robustness criterion of

$$|\lambda_* - \lambda_0| \geq 2.0$$

where λ_* is the closest bifurcation to λ_0.

Design Strategy

Calculate the closest bifurcation λ_{0*}^0 to λ_0^0. If

$$\left|\lambda_{0*}^0 - \lambda_0^0\right| \geq 2.0$$

then $\lambda_0 = \lambda_0^0$ and the problem is solved. But if

$$\left|\lambda_{0*}^0 - \lambda_0^0\right| < 2.0$$

then obtain a new design via the formula

$$\lambda_0^1 = \lambda_0^0 - \left(2.0 - \left|\lambda_{0*}^0 - \lambda_0^0\right|\right)\frac{\lambda_{0*}^0 - \lambda_0^0}{\left|\lambda_{0*}^0 - \lambda_0^0\right|} \tag{31}$$

Then calculate the closest bifurcation point λ_{0*}^1 to λ_0^1 and check whether or not

$$\left|\lambda_{0*}^1 - \lambda_0^1\right| \geq 2.0$$

If it is, then $\lambda_0 = \lambda_0^1$ and the problem is solved; if not, then repeat the preceding steps until the robustness criterion is satisfied.

Results

Proceeding as described above, the closest bifurcation point to λ_0^0 is

$$\lambda_{0*}^0 = (3.99285, 4.00000, 0.05009, 3.00076, 5.68917, 6.46700)$$

Checking $|\lambda_{0*}^0 - \lambda_0^0|$, we find that

$$\left|\lambda_{0*}^0 - \lambda_0^0\right| = 0.66203 < 2.0$$

and Eq. (31) yields

$$\lambda_0^1 = (4.01584, 4.00000, 0.04980, 2.99832, 7.01010, 4.96547)$$

Calculating the closest bifurcation to λ_0^1, we find

$$\lambda_{0*}^1 = (3.99285, 4.00000, 0.05009, 3.00076, 5.68917, 6.46700)$$

and

$$\left|\lambda_{0*}^1 - \lambda_0^1\right| = 2.0$$

Therefore

$$\lambda_0 = \lambda_0^1 = (4.01584, 4.00000, 0.04980, 2.99832, 7.01010, 4.96547)$$

and a design satisfying the robustness criterion has been achieved.

Summary

We consider designing the robust stability of Hamiltonian satellites with multiple design parameters and an equilibrium fixed at the origin. The origin can become unstable in a transcritical or pitchfork bifurcation, and in designing the satellite parameters we seek to avoid the critical parameters Σ at which the bifurcation occurs. Given nominal design parameters λ_0, we compute the parameters λ_* of a bifurcation locally closest in parameter space to λ_0 and use the stability margin $|\lambda_* - \lambda_0|$ to check the robustness of the stability of the design at λ_0. The computation of the closest bifurcation is considerably simplified by the Hamiltonian assumption. The standard computation of the bifurcation in a given direction of parameter increase from λ_0 is also simplified. If the stability margin $|\lambda_* - \lambda_0|$ is too small, we adjust the nominal design λ_0 to increase its robustness to the desired amount. This adjustment uses formulas for the sensitivity of $|\lambda_* - \lambda_0|$ appropriately simplified by the Hamiltonian assumption. The closest bifurcation methods of this paper apply more generally to Hamiltonian systems with additional damping terms provided these terms have no linear dependence on position. To illustrate the methods, we consider the design of a flexible dual-spin satellite with six design parameters to achieve a given stability margin in the six-dimensional parameter space.

Appendix

The equations $\dot{x} = f(x, \lambda)$ are assumed to have a generic transcritical or pitchfork bifurcation at $\lambda_* \in \Sigma$, and we state a standard argument yielding the normal vector formula (3). It is then straightforward to indicate the modifications to obtain the normal vector formula (19) for the Hamiltonian system.

Since the zero eigenvalue of Df is unique, there is a smooth function μ' defined in a neighborhood of λ_* with $\mu'(\lambda_*) = 0$ and $\mu'(\lambda)$ the eigenvalue of $Df|_{(0,\lambda)}$ with smallest absolute value. Write v', w' for the right and left eigenvectors of $Df|_{(0,\lambda)}$ corresponding to $\mu'(\lambda)$; these eigenvectors are normalized according to $w'v' = 1$. Then v' and w' are smooth functions of λ in a neighborhood of λ_*. Write $v'_* = v'(\lambda_*)$ and $w'_* = w'(\lambda_*)$. Here, Σ is given by $\mu'(\lambda) = 0$ in a neighborhood of λ_* so that the normal vector $N(\lambda_*)$ is given by the gradient $D_\lambda \mu'|_{\lambda_*}$, which we compute by differentiating,

$$\mu' = w' Df|_{(0,\lambda)} v' \tag{A1}$$

with respect to λ to obtain

$$\begin{aligned} N(\lambda_*) &= w'_*(D_{x\lambda} f|_{(0,\lambda_*)})v'_* + (D_\lambda w'|_{\lambda_*})Df|_{(0,\lambda_*)}v'_* \\ &\quad + w'_* Df|_{(0,\lambda_*)}(D_\lambda v'|_{\lambda_*}) \\ &= w'_*(D_{x\lambda} f|_{(0,\lambda_*)})v'_* \end{aligned} \tag{A2}$$

To derive the analogous formula for the Hamiltonian case, we first need to demonstrate that the zero eigenvalue of $D_q^2 h_0$ is unique. The eigenvector v' of Df corresponding to the unique zero eigenvalue of Df is also an eigenvector of D^2H corresponding to a zero eigenvalue of D^2H since $D^2H v' = JDfv' = 0$. Moreover, if v'' is any eigenvector corresponding to a zero eigenvalue of D^2H, then $Dfv'' = JD^2Hv'' = 0$ so that v'' is a scalar multiple of v' by the uniqueness of the zero eigenspace of Df. Therefore the zero eigenvalue of D^2H is unique. Since D^2H and D^2h have the same inertia [cf. Eq. (13)], the zero eigenvalue of D^2h is also unique. The positive definiteness of $D^2_{q\mathrm{dot}} h_2$ then implies that the zero eigenvalue of $D_q^2 h_0$ is unique.

The uniqueness of the zero eigenvalue of $D_q^2 h_0(\lambda_*)$ implies that we can define a smooth function μ in a neighborhood of λ_* with $\mu(\lambda_*) = 0$ and $\mu(\lambda)$ the eigenvalue of $D_q^2 h_0(\lambda)$ with smallest absolute value. Write v for the eigenvector of $D_q^2 h_0(\lambda)$ corresponding to $\mu(\lambda)$. Here, Σ is given by $\mu(\lambda) = 0$ in a neighborhood of λ_* so that the normal vector $N(\lambda_*)$ may be computed by differentiating,

$$\mu = v^T D_q^2 h_0(\lambda) v \tag{A3}$$

to obtain

$$N(\lambda_*) = v_*^T D_\lambda D_q^2 h_0(\lambda_*) v_* \tag{A4}$$

by reproducing the steps above.

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